Approximation Algorithms **Homework 1**
Date: Sept 14, 2005 **Due:** Sept 21, 2005

1. K**-center:** Consider an undirected *complete* graph $G = (V, E)$ with nonnegative *metric* distances d on the edges; i.e., $d_{ij} + d_{jk} \geq d_{ik}$ for all $i, j, k \in V$. Given G and a positive integer K, the K-center problem asks for a subset $C \subseteq V$ of vertices of size $|C| = K$ that minimizes

$$\max_{v \in V} d(v, C). \tag{1}$$

Let D^* be this minimum value. (Note here that $d(v, C) = \min_{x \in C} d(v, x)$ is the natural extension of the distance function to sets, namely, it is the distance of vertex v to the closest node in C.)

Before you begin, convince yourself that this problem is NP-hard. (You don't have to write this down.) Our goal is to develop an approximation algorithm for D^*.

(a) A natural *greedy* 2-approximation algorithm for this problem first assumes that we have the right guess for D^*. it starts with all nodes being *unmarked*, iteratively chooses some unmarked node as a center, marks all the nodes within a distance $2D^*$ of this chosen node as covered, and continues the process with the remaining unmarked nodes until all nodes are marked. Clearly, each node is within a distance $2D^*$ of some chosen node at the end of the algorithm: to complete the argument, show that we cannot have chosen more than K centers before all nodes are marked.

(b) Given an instance of the K-center problem, show that D^* can only take one of $\binom{n}{2}$ values (which depend on the given instance, of course.) In particular, this will give a method to enumerate set of $O(n^2)$ values for any given instance, one of which must be D^*. How can this allow you to remove the assumption that we had the right guess for D^*?

(c) Find an example (actually even better, an infinite family of examples parameterized by size) that demonstrates that the above greedy algorithm will produce a solution of value $2D^*$ and no better on the example.

2. K**-dispersion:** Given an undirected complete graph $G = (V, E)$ with nonnegative *metric* distances d on the edges and a given positive integer K, the K-dispersion problem asks for a subset $S \subseteq V$ of vertices of size k (i.e., $|S| = k$) that are as far apart from each other as possible. Namely, we are trying to *maximize*

$$\min_{s, s' \in S} d(s, s') \tag{2}$$

Let this quantity be denoted by F^*. A typical application of this problem is to locate harmful goods (like nuclear facilities, waste dumps etc.) far apart to minimize collective harm that might result from interaction.

Consider the following greedy algorithm: start with S being an arbitrary node from V, and repeatedly add a node furthest from the nodes currently in S, until you have a total of k

nodes. Show that this is a 2-approximation algorithm for F^*, and show examples to show that your algorithm has a performance guarantee no better than 2.

To strengthen the last part on the limits to the performance of the greedy algorithm, we can try to prove a inapproximability result as well. Show that, unless P = NP, there is no 1.9999-approximation algorithm for K-dispersion.

3. Prove the following *tree-pairing lemma* analogous to the *tour-pairing lemma* given in Lecture:

   Lemma 1 (Tree-Pairing Lemma) *Given an undirected tree $T = (V, E)$ and a subset $M \subseteq V$ of even cardinality, find a pairing (perfect matching) of the nodes in M such that the paths in T between the matched pairs are edge-disjoint.*

4. Classify the following problems into P or NP. Provide a brief explanation with each.

   (a) Given an undirected graph $G = (V, E)$ with nonnegative costs c on the edges, a *target degree* d_v for each node $v \in V$, find a subgraph (subset of edges) of minimum total cost whose degree at node v is d_v.

   (b) Given an undirected graph $G = (V, E)$ with nonnegative costs c on the edges, a *degree lower bound* ℓ_v for each node $v \in V$, find a subgraph (subset of edges) of minimum total cost whose degree at node v is at least ℓ_v.

   (c) Given an undirected graph $G = (V, E)$ with nonnegative costs c on the edges, a *degree upper bound* u_v for each node $v \in V$, find a subgraph (subset of edges) of minimum total cost whose degree at node v is at most u_v.

   (d) Given an undirected graph $G = (V, E)$ with nonnegative costs c on the edges, and a vertex $v_0 \in V$, find a walk (i.e., a sequence of consecutive edges) of minimum total cost such that the walk starts and ends at v_0, and every edge of the graph appears at least once in the walk.